

Experimental finding of modular forms for noncongruence subgroups

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October 5, 2009

Abstract

In this paper we will use experimental and computational methods to find modular forms for non-congruence subgroups, and the modular forms for congruence subgroups that they are associated with via the Atkin–Swinnerton-Dyer correspondence. We also prove a generalization of a criterion due to Ligozat for an eta-quotient to be a modular function.

1 Introduction

Let N be a positive integer. We define $\Gamma(N)$ to be the group of invertible 2×2 matrices with coefficients in \mathbf{Z} whose reduction modulo N is congruent to the identity matrix. We say that a subgroup of the “modular group” $\mathrm{SL}_2(\mathbf{Z})$ is a *congruence subgroup* if it contains $\Gamma(N)$ for any N . It can be shown that these subgroups have finite index in $\mathrm{SL}_2(\mathbf{Z})$. We define a *non-congruence subgroup* to be a subgroup of finite index inside $\mathrm{SL}_2(\mathbf{Z})$ which is not a congruence subgroup.

The theory of modular forms for congruence subgroups is well-established, at least in integral weights; there are algorithms to compute bases of space of modular forms, and a well-understood arithmetic theory of Hecke operators acting on these spaces. There are many good introductions to this; see [18] or [4], for example.

However, as [1] says in its introduction, the theory of modular forms for non-congruence subgroups is much less well-known, despite the fact that (in a sense that can be made rigorous) most subgroups of the modular group of finite index are not congruence subgroups (see [7] for a more precise statement of this result).

The pioneering experimental work of [2] discovered congruences satisfied by certain modular forms of this type, which are now known as Atkin–Swinnerton-Dyer congruences, and certain of these have been proved by Scholl in a series of papers [14–17], which also consider the Hecke algebras attached to spaces of modular forms for non-congruence subgroups.

More recently, there has been work on refining the conjectured congruences by Atkin, Li, Long and Yang; see [1, 10, 11]. They prove the Atkin–Swinnerton-Dyer congruences for certain specific cases, and give another version of the con-

jectures made earlier in the field. They also show that the L-functions attached to certain non-congruence modular forms by Scholl are “modular”, in the sense that they can be attached to modular forms for congruence subgroups.

Experimentally, it has been noted by Atkin and Swinnerton-Dyer and others that the denominators of modular forms for non-congruence subgroups are unbounded; this is in sharp contrast to the situation for modular forms for congruence subgroups, which are well-known to have bounded denominators. It is an interesting open question whether all modular forms for non-congruence subgroups have this unbounded denominator property; recently [9] considered this question; they prove the unbounded denominator property for certain non-congruence subgroups.

There has also been recent computational work by Verrill et al [5], who have found a number of new examples of modular forms for non-congruence subgroups which are conjectured to satisfy the Atkin–Swinnerton-Dyer congruences. This paper inspired the current work, which also gives lists of modular forms for non-congruence subgroups which are conjectured to satisfy Atkin–Swinnerton-Dyer congruences.

The computational work referred to above found congruences involving modular forms for non-congruence subgroups of genus 0. In Richards [13], algorithms are given which extend this to general non-congruence subgroups, and explicit examples for genus 1 groups are exhibited. This work is particularly interesting because it uses complex approximations to modular forms rather than p -adic approximations, thus giving a different and unusual perspective on the subject.

2 Notation

We first give an explicit description of the Atkin–Swinnerton-Dyer congruence relation, following that given in [11]. We suppose that Γ is a non-congruence subgroup of finite index in the modular group $\mathrm{SL}_2(\mathbf{Z})$, and that k is a non-negative integer.

Definition 1. *Suppose that Γ has cusp width μ at infinity, and that $h \in S_k(\Gamma)$ has an M -integral Fourier expansion at infinity in terms of $q^{1/\mu}$ of the form*

$$h(q) = \sum_{n=1}^{\infty} a_n q^{n/\mu},$$

for some integer M .

Let f be a normalized newform of weight k , level N and character χ (for some congruence subgroup) with Fourier expansion at infinity given by

$$f(q) = \sum_{n=1}^{\infty} c_n q^n.$$

We say that the forms f and h satisfy the Atkin–Swinnerton-Dyer congruence relation if, for all primes p not dividing MN and for all positive integers n , we

have that

$$\frac{a_{np} - c_p a_n + \chi(p) p^{k-1} a_{n/p}}{(np)^{k-1}} \quad (1)$$

is integral at all places dividing p . We define $a_{n/p}$ to be zero if $p \nmid n$.

This is modelled upon the following well-known recurrence relation that holds for the Fourier coefficients of modular forms for congruence subgroups which are normalized simultaneous eigenvectors for the Hecke operators:

$$a_{np} - a_p a_n + \chi(p) p^{k-1} a_{n/p} = 0,$$

where again we take $a_{n/p}$ to be zero if $p \nmid n$.

Again inspired by the existence of a basis of normalized eigenforms for spaces of newforms in the congruence case, we now define an Atkin–Swinnerton-Dyer basis.

Definition 2. Let k be a non-negative integer and let Γ be a non-congruence subgroup. We say that $S_k(\Gamma)$ has an Atkin–Swinnerton-Dyer basis if for every prime number p there is a basis $\{h_1, \dots, h_n\}$ of $S_k(\Gamma)$ and normalized newforms f_1, \dots, f_n such that each pair (h_i, f_i) satisfies the Atkin–Swinnerton-Dyer congruence relation given in (1).

We note that there are cases where one choice of Atkin–Swinnerton-Dyer basis will suffice for all but finitely many primes p , and others where the basis depends on the value of p modulo some integer N . We will describe these below.

3 Extending the Ligozat criterion

First, we recall the definition of the Dedekind η function; let z be an element of the Poincaré upper half plane. Then we have

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \text{ where } q := \exp(2\pi iz).$$

The η -function can be used to build many interesting modular forms; for instance, the Δ -function is the 24th power of η , and in [8] it is proved that every modular form for certain congruence subgroups can be written as a sum of η -quotients.

We will prove a generalization of the criterion of Ligozat given in Section 3 of [12] for an η -quotient to be a modular function with character for $\Gamma_0(N)$.

We recall that, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, with $c \geq 0$, then

$$\eta\left(\frac{az+b}{cz+d}\right) = \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (-i(cz+d))^{1/2} \cdot \eta(z), \quad (2)$$

where

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \exp\left(-i\pi\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \text{ and } \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{Z}.$$

The actual definition of α is rather complicated (it involves Dedekind sums; for more details, see Section 2.8 of [6] for the full story, for instance), but if we have $(a, 6) = 1$, then the following congruence holds:

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \frac{1}{12} \cdot a(c - b - 3) - \frac{1}{2} \left(1 - \left(\frac{c}{a} \right) \right) \pmod{2}.$$

This is very useful because it can be shown that $\Gamma_0(N)$ can be generated by matrices of the form

$$\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N), \text{ with } (a, 6) = 1 \text{ and } a, c \geq 0,$$

so we need only verify the transformation condition on matrices of this form to prove our theorem.

Let N be a positive integer and define $g(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$.

Theorem 3. *Let N be a positive integer and let $g(z)$ be as defined above. Suppose that:*

1. $\sum_{\delta|N} r_\delta \cdot \delta \equiv 0 \pmod{24}$,
2. $\sum_{\delta|N} r_\delta \cdot (N/\delta) \equiv 0 \pmod{24}$ and
3. $\sum_{\delta|N} r_\delta = 0$.

Then $g(z)$ is a modular function of weight 0 for $\Gamma_0(N)$ with quadratic character $\chi := \prod_{\delta|N} \left(\frac{N/\delta}{\cdot} \right)^{r_\delta}$.

Proof. We take $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and δ to be a divisor of N . By explicit computation, we see that

$$\eta(\delta U z) = \eta(U_\delta \cdot \delta z) \text{ where } U_\delta = \begin{pmatrix} a & b\delta \\ c\delta' & d \end{pmatrix} \text{ with } \delta \cdot \delta' = N.$$

Using the explicit formula for the transformation of η given in (2), we see that we have

$$g(Uz) = (-i(Ncz + d))^{\sum_{\delta|N} \frac{r_\delta}{2}} \cdot g(z) \cdot \prod_{\delta|N} \varepsilon(U_\delta)^{r_\delta}.$$

From assumption (3) of the theorem, we see that the first factor vanishes, so we now need to evaluate the third factor. of the cases that we are considering in the theorem we will have $(a, 6) = 1$ (either we have a generator of $\Gamma_0(N)$ in which case we can assume this, or we have the auxiliary level structure $\Gamma(3)$ which will also allow us to assume this), so we can rewrite the third factor as

$$\prod_{\delta|N} \varepsilon(U_\delta)^{r_\delta} = \exp(-i\pi\lambda) \text{ where } \lambda = \sum_{\delta|N} r_\delta \cdot \alpha(U_\delta).$$

Now using the fact that $(a, 6) = 1$ (because we are dealing with a generator of $\Gamma_0(N)$) we can write out $\alpha(U_\delta)$ explicitly as

$$\alpha(U_\delta) \equiv \frac{1}{12}a(c\delta' - b\delta - 3) - \frac{1}{2} \left(1 - \left(\frac{c\delta'}{a} \right) \right) \pmod{2},$$

which means that we can write λ modulo 2 as

$$\lambda \equiv \frac{1}{12} \left(\sum_{\delta|N} r_\delta \cdot \delta' \right) - \frac{1}{12}ab \left(\sum_{\delta|N} r_\delta \cdot \delta \right) - \frac{a}{4} \sum_{\delta|N} r_\delta - \frac{1}{2} \sum_{\delta|N} \left[1 - \left(\frac{c\delta'}{a} \right) \right] \cdot r_\delta.$$

We now use the fact that the sum of the degrees r_δ is 0 to show that the third term of the right-hand side is 0. As the congruences in (1) and (2) hold modulo 24, the first and second terms will vanish modulo 2. that our matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(3)$ and in particular that $b \equiv c \equiv 0 \pmod{3}$ to show that the first and second terms in the congruence for λ are integral and still vanish modulo 2.

This means that λ in fact satisfies the congruence

$$\lambda \equiv \frac{1}{2} \sum_{\delta|N} \left[1 - \left(\frac{c\delta'}{a} \right) \right] \cdot r_\delta \pmod{2},$$

and therefore that we have

$$\exp(-i\pi\lambda) = \prod_{\delta|N} \left(\frac{\delta'c}{a} \right)^{r_\delta} = \prod_{\delta|N} \left(\frac{\delta'}{a} \right)^{r_\delta};$$

where we can take out the factor of c using (3). This means that we can rewrite $g(Uz)$ as

$$g(Uz) = \prod_{\delta|N} \left(\frac{\delta'}{a} \right)^{r_\delta} g(z),$$

so we have shown that g transforms correctly under the action of elements of $\Gamma(N)$, which proves our theorem. \square

The proof will also go through if we take the congruences in (1) and (2) modulo 8; in that case, the proof shows that g is a modular function for the congruence subgroup $\Gamma_0(N) \cap \Gamma(6)$. We cannot expect to prove that it is a modular function for $\Gamma_0(N)$ because its Fourier expansion is given in terms of $q^{1/3}$ and not q .

We note also that if all of the r_δ are even, then the fact that $\eta(q)^2$ generates the space of modular forms $S_1(\Gamma(12))$ implies that g is a modular function for the congruence subgroup $\Gamma(12N)$.

4 Algorithm used for finding modular forms for non-congruence subgroups

The method that we use here is basically a converse to that described in [5]. We consider roots of η -quotients of the form

$$\sqrt[3]{\eta(q^a)^m \eta(q^b)^n \eta(q^c)^r \eta(q^d)^s}, \quad (3)$$

where the a, b, c, d are positive integers which divide either 6 or 8, and $m + n + r + s = 18$ (so the modular function given in (3) has weight 3). We assume the unbounded denominator question discussed in the introduction, that modular forms for non-congruence subgroups have unbounded denominator, to speed up the calculations. We cannot use our extension of a theorem of Ligozat here, because it deals with η -quotients rather than their roots.

We now consider the specific situation where p is a prime, the weight k is 3, n is a positive integer not divisible by p , and we have an Atkin–Swinnerton–Dyer basis (with respect to p) of our space of modular forms for a noncongruence subgroup which is composed of η -quotients (this is called “Case 1” in [5]). In this particular case, the equation (1) reduces to

$$\frac{a_{np} - c_p a_n}{(np)^2},$$

so in particular we see that $a_{pn} \equiv c_p a_n \pmod{p^2}$, and (if $a_n \neq 0$) then we have $a_{np}/a_n \equiv c_p \pmod{p^2}$. If we now fix p and let n vary, then the term on the right hand side of our equation will remain constant, as it does not depend on n , so we have

$$a_p \equiv c_p \pmod{p}, \quad (4)$$

as long as all of the terms that we have been manipulating were nonzero modulo p^2 . It may happen that we have to consider a twist $f \otimes \chi$ of f to actually get congruences for every prime; this we can detect by checking to see if c_p/a_p is a root of unity for all of the primes p .

However, for some primes the Atkin–Swinnerton–Dyer basis will not be a pair of distinct η -quotients h_1 and h_2 , but will instead be of the form $h_1 + \alpha h_2$, where α is an algebraic number of small degree (this is called “Case 2” in [5]). We assume that this means that we have

$$a_{pn} + \alpha b_{pn} \equiv c_p(a_n + \alpha b_n) \pmod{p^2}, \quad (5)$$

where as above the p^2 comes from (1) with $k = 3$. This will hold if

$$a_{pn} \equiv c_p a_n \pmod{p^2} \text{ and } \alpha b_{pn} \equiv c_p \alpha b_n \pmod{p^2}, \quad (6)$$

and if every term here is nonzero modulo p^2 then this implies that $a_{pn}/b_n \equiv c_p \alpha$ modulo p^2 and $b_{pn}/a_n \equiv c_p/\alpha$ modulo p^2 . As above, we see that the right-hand side of these congruences do not depend on n , so if we fix p and vary n then we will find that both a_{pn}/b_n and b_{pn}/a_n are constant modulo p^2 .

We can use these congruences to find α^2 and c_p^2 modulo p^2 by combining the congruences above; we find (assuming that the terms are nonzero) that

$$\alpha^2 \equiv \frac{\frac{a_{np}}{b_n}}{\frac{b_{np}}{a_n}} \text{ and } c_p^2 \equiv \frac{a_{np}}{b_n} \cdot \frac{b_{np}}{a_n}; \quad (7)$$

both of these quantities are (at least experimentally) well-defined because we have shown that the terms are constant modulo p^2 . As in the previous case, we may need to consider a twist of the form f by a character χ .

We will use this in the following way; we will run over all η -quotients of the form (3) where m, n, r, s are less than some bound, and find pairs of η -quotients which satisfy one of the two cases described above for each prime p up to a specified bound. The calculations here were performed using MAGMA [3]; other computer algebra packages such as SAGE [19] would also be suitable for this.

Given an experimentally found η -quotient, we would like to show that this is a modular form. We will do this by writing it as the product of a known modular form for a non-congruence subgroup and a modular function of weight 0 for a congruence subgroup, which will show that it is a modular *function* for the intersection of these groups, and then we will verify that its cube is a modular form, so it has no poles on the upper half plane and therefore is a modular *form*.

The final part of the puzzle is to identify the modular form f for a congruence subgroup which satisfies an Atkin–Swinnerton-Dyer congruence with our η -quotient. This is mostly a matter of trial and error; one can guess that the level will be divisible by the primes 2 and 3, and using MAGMA we can compute spaces of modular forms of weight 3 for congruence subgroups. We also have some idea of what the coefficients of f should be, because we can use (4) and (7) to work out what those coefficients or their squares are modulo p^2 .

5 Tables of results

We list some modular forms, mostly taken from Table 12 of [5] for $\Gamma_1(6)$ and $\Gamma_1(12)$, which we can use as building blocks for our non-congruence modular forms.

$$\begin{aligned} a &= \frac{\eta(q)\eta(q^6)^6}{\eta(q^2)^2\eta(q^3)^3} = q - q^2 + q^3 + q^4 + \dots \\ b &= \frac{\eta(q^2)\eta(q^3)^6}{\eta(q)^2\eta(q^6)^3} = 1 + 2q + 4q^2 + 2q^3 + \dots \\ c &= \frac{\eta(q^2)^6\eta(q^3)}{\eta(q)^3\eta(q^6)^2} = 1 + 3q + 3q^2 + 3q^3 + \dots \\ d &= \frac{\eta(q)^6\eta(q^6)}{\eta(q^2)^3\eta(q^3)^2} = 1 - 6q + 12q^2 - 6q^3 + \dots \\ e &= \frac{\eta(q)^2\eta(q^3)^2}{\eta(q^2)\eta(q^6)} = 1 - 2q - 2q^3 + \dots \end{aligned}$$

We follow the notation of Verrill et al for these forms; all of them apart from e are modular forms of weight 1 for $\Gamma_1(6)$, and e is a modular form of weight 1 for $\Gamma_1(12)$.

Similarly, there are modular forms and functions listed in Table 11 of [5] for $\Gamma_1(4) \cap \Gamma_0(8)$ and $\Gamma_1(16)$ which we can use to construct non-congruence modular forms. Again, we follow the notation given in [5].

$$\begin{aligned}
t &= \frac{\eta(q)^8 \eta(q^4)^4}{\eta(q^2)^{12}} \in M_0(\Gamma_1(4) \cap \Gamma_0(8)) \\
\frac{t+1}{2} &= \frac{\eta(q)^4 \eta(q^4)^{14}}{\eta(q^2)^{14} \eta(q^8)^4} \in M_0(\Gamma_1(4) \cap \Gamma_0(8)) \\
\frac{t+1}{2t} &= \frac{\eta(q^4)^{10}}{\eta(q)^4 \eta(q^2)^2 \eta(q^8)^4} \in M_0(\Gamma_1(4) \cap \Gamma_0(8)) \\
\frac{4(t+1)}{1-t} &= \frac{\eta(q^4)^{12}}{\eta(q^2)^4 \eta(q^8)^8} \in M_0(\Gamma_1(4) \cap \Gamma_0(8)) \\
\sqrt{t} &= \frac{\eta(q)^4 \eta(q^4)^2}{\eta(q^2)^6} \in M_0(\Gamma_1(16)) \\
\sqrt{\frac{t+1}{2}} &= \frac{\eta(q)^2 \eta(q^4)^7}{\eta(q^2)^7 \eta(q^8)^2} \in M_0(\Gamma_1(16)) \\
E_a &= \frac{\eta(q^2)^6 \eta(q^4)^4}{\eta(q)^4} \in M_3(\Gamma_1(4) \cap \Gamma_0(8)) \\
E_b &= \left(\frac{2t}{t+1} \right) E_a = \frac{\eta(q^2)^8 \eta(q^8)^4}{\eta(q^4)^6} \in M_3(\Gamma_1(4) \cap \Gamma_0(8)).
\end{aligned}$$

Firstly, we present two tables of forms listed in Tables 13 and 14 of [5], which have been shown to be modular forms for certain explicit non-congruence subgroups contained within $\Gamma_1(4) \cap \Gamma_0(8)$ and $\Gamma_1(6)$.

h_1		h_2	
$[4, 7, -4, 11]$	$\left(\sqrt[3]{b/d} \right) acd$	$[-4, 11, 4, 7]$	$\left(\sqrt[3]{b/d} \right)^2 acd$
$[13, -2, -7, 14]$	$\left(\sqrt[3]{b/c} \right) acd$	$[14, -7, -2, 13]$	$\left(\sqrt[3]{b/c} \right)^2 acd$

Figure 1: Forms for subgroups of $\Gamma_1(12)$ from Table 14 of [5].

We now present two tables of pairs of noncongruence forms which form Atkin–Swinnerton-Dyer bases that we have found experimentally. We represent the η quotient given in (3) by the tuple $[a, b, c, d]$; if there is one form given twice, then $\{h_1, h_2\}$ forms an Atkin–Swinnerton-Dyer basis for all but finitely many primes p , whereas if there are two distinct forms given then they form an Atkin–Swinnerton-Dyer basis of the form $\{h_1 \pm \alpha h_2\}$ for primes satisfying a congruence condition.

h_1		h_2	
$[-8, 20, 2, 4]$	$\sqrt[3]{\frac{t+1}{2}} E_b$	$[-4, 22, -8, 8]$	$\sqrt[3]{\frac{t+1}{2}}^2 E_b$
$[-4, 6, 16, 0]$	$t^{1/3} E_a$	$[4, -6, 20, 0]$	$t^{2/3} E_a$
$[4, 10, -4, 8]$	$\sqrt[3]{\frac{t+1}{2t}} E_b$	$[8, -4, 10, 4]$	$\sqrt[3]{\frac{t+1}{2t}}^2 E_b$
$[0, 20, -6, 4]$	$\sqrt[3]{\frac{4(t+1)}{1-t}} E_b$	$[0, 16, 6, -4]$	$\sqrt[3]{\frac{4(t+1)}{1-t}}^2 E_b$

Figure 2: Forms for subgroups of $\Gamma_0(8) \cap \Gamma_1(4)$ from Table 13 of [5].

We see that the new non-congruence modular forms we have discovered can be written as products of modular forms in the same way as those found by Verrill et al, so we can think of these forms we have found as fitting into the same framework as those in [5].

h_1		h_2	
$[-8, 13, 8, 5]$	$\left(\sqrt[3]{e/b}\right) abc$	$[8, 5, -8, 13]$	$\left(\sqrt[3]{e/b}\right)^2 ace$

Figure 3: Forms for subgroups of $\Gamma_1(12)$ which form AS-D bases

h_1		h_2	
$[-2, 23, -13, 10]$	$\left(\frac{t+1}{2t}\right)^{1/6} E_b$	$[-10, 19, 7, 2]$	$\left(\frac{t+1}{2t}\right)^{5/6} E_b$
$[8, -12, 22, 0]$	$\sqrt{t} E_a$	$[-8, 12, 14, 0]$	$\sqrt{t}^5 E_a$
$[0, -8, 30, -4]$	$\left(\frac{4(t+1)}{1-t}\right)^{8/3} E_b$	$[0, 8, 6, 4]$	$t^{-2/3} E_b$
$[2, 17, -11, 10]$	$\left(\frac{t+1}{2}\right)^{1/6} E_a$	$[10, -11, 17, 2]$	$\left(\frac{t+1}{2}\right)^{5/6} E_a$

Figure 4: Forms for subgroups of $\Gamma_0(16) \cap \Gamma_1(4)$ which form AS-D bases

To show that the new examples we have found are modular forms we check that we can write them as products of the form $f \cdot g$, where f is a known weight 3 modular form for a noncongruence subgroup (such as those from Verrill et al's tables), and g is a modular function of weight 0 for a congruence subgroup, and that $(f \cdot g)^3$ is a modular form. In Section 6 we give a fully worked-out example of this.

We also give more detail on the forms related by Atkin–Swinnerton-Dyer congruences; their Fourier expansions at ∞ and the characters of the congruence forms associated to them.

6 Worked example

We will now illustrate how we find two modular forms for a non-congruence subgroup Γ contained in $\Gamma_0(16)$. We first run a search for η -quotients of the form $\sqrt[3]{\eta(q)^m \eta(q^2)^n \eta(q^4)^r \eta(q^8)^s}$ which experimentally satisfy the congruence conditions discussed after (6), and we find the following two examples which do not appear in the tables of [5]:

$$H_1 := \sqrt[3]{\frac{\eta(q^2)^{12} \eta(q^4)^{14}}{\eta(q)^8}} \text{ and } H_2 := \sqrt[3]{\frac{\eta(q)^8 \eta(q^4)^{22}}{\eta(q^2)^{12}}}.$$

We now notice that we can write H_1 and H_2 as products of a modular function for $\Gamma_0(16)$ and h_1 and h_2 from the second row of Figure 2; we have that

$$H_1 = \frac{\eta(q^2)^6}{\eta(q)^4 \eta(q^4)^2} \cdot h_2 \text{ and } H_2 = \frac{\eta(q)^4 \eta(q^4)^2}{\eta(q^2)^6} \cdot h_1.$$

We now show that the η -quotients here are actually modular functions for $\Gamma_0(16)$ using Theorem 3. We verify easily that the η -quotient given above satisfies this for $N = 16$ although not for $N = 8$, which shows that H_1 and H_2 are modular functions of weight 3 for a suitable noncongruence subgroup. Using MAGMA and SAGE it can be checked that $H_1^3, H_2^3 \in S_9(\Gamma_1(16))$, which verifies that both H_1 and H_2 are in fact modular forms.

Finally, we will experimentally determine a classical modular form which satisfies an Atkin–Swinnerton-Dyer congruence. Figure 5 is a table which shows the values of a_{np}/a_n and b_{np}/b_n modulo p^2 , where an empty space indicates that these numbers are not constant modulo p^2 .

p	$a_{np}/a_n, b_{np}/b_n$	$a_{np}/b_n, b_{np}/a_n$
5	6	
7		0
11		0
13		10
17	30	
19		0
23		0
29		
31	-42	0
37		-70
41		
43		0
47	-18	0

Figure 5: Experimentally computed values of a_{np}/a_n and b_{np}/b_n modulo p^2 , where we take primes $p \geq 5$ and positive integers n with $pn \leq 500$.

Let τ be the nontrivial character modulo 4 and let $f \in S_3(\Gamma_0(144), \tau)$ be the unique normalized new eigenform with Fourier expansion at ∞ beginning $f(q) = q + 6q^5 + 10q^{13} + O(q^{17})$; this is the twist of the level 16 η -product $\eta(q^4)^6$ by the Legendre character modulo 3 (f was found by noticing that the η -product almost satisfied the congruence conditions, and then working out which twist actually worked). Here is the Fourier expansion at ∞ of f up to $O(q^{50})$:

$$q + 6q^5 + 10q^{13} + 30q^{17} + 11q^{25} - 42q^{29} - 70q^{37} - 18q^{41} + 49q^{49} + O(q^{50}).$$

It can be seen that the Fourier coefficients of f are congruent modulo p^2 to those given in Figure 5.

We conjecture that the Atkin–Swinerton-Dyer basis is $\{H_1, H_2\}$ if $p \not\equiv 5 \pmod{12}$, and that it is $\{H_1 + H_2, H_1 + (p^2 + 1)H_2\}$ when $p \equiv 5 \pmod{12}$.

7 Acknowledgements

I would like to thank Helena Verrill and Ken McMurdy for helpful conversations, and the Heilbronn Institute for its support while I was writing this paper. Some of the computations were performed on William Stein’s computer `sage` and I would like to thank him for the use of this machine.

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